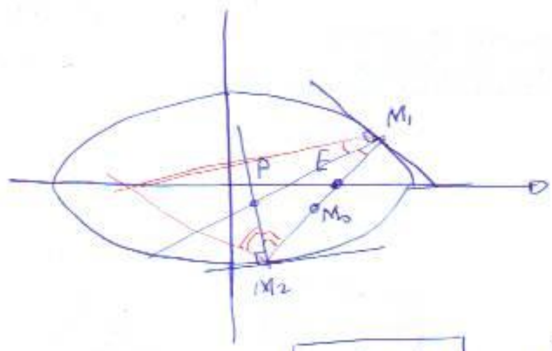
ΠΡΟΤΕΙΝΟΜΕΝΗ ΑΣΚΗΣΗ 2 ΕΥΚΛΕΙΔΗΣ Β' – ΤΕΥΧΟΣ 92

ΘΕΜΑ

Δίνεται έλλειψη C με εστίες E, E' στον άξονα $x'x$, δύο τυχαία σημεία της M_1, M_2 συνευθειακά με την εστία E, M_0 το μέσον της χορδής M_1M_2 και P το σημείο τομής των ευθειών u_1, u_2 που είναι κάθετες στις εφαπτόμενες της έλλειψης στα σημεία M_1, M_2 αντιστοίχως. να αποδειχθεί ότι η ευθεία PM_0 είναι παράλληλη στον άξονα $x'x$.

ΛΥΣΗ



$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$$

$$\beta^2x^2+\alpha^2y^2-\alpha^2\beta^2=0$$

$$y-0=\lambda(x-\gamma) \Leftarrow \boxed{y=\lambda x-\lambda\gamma}$$

$$\beta^2x^2+\alpha^2(\lambda^2x^2+\lambda^2\gamma^2-2\lambda^2\gamma x)-\alpha^2\beta^2=0$$

$$(\beta^2+\alpha^2\lambda^2)x^2-2\lambda^2\alpha^2\gamma x+\alpha^2\lambda^2\gamma^2-\alpha^2\beta^2=0$$

$$x_1+x_2=\frac{2\lambda^2\alpha^2\gamma}{\beta^2+\alpha^2\lambda^2} \qquad x_1\cdot x_2=\frac{\alpha^2\lambda^2\gamma^2-\alpha^2\beta^2}{\beta^2+\alpha^2\lambda^2}$$

$$y_1y_2=(\lambda x_1-\lambda\gamma)(\lambda x_2-\lambda\gamma)=\lambda^2\frac{\alpha^2\lambda^2\gamma^2-\alpha^2\beta^2}{\beta^2+\alpha^2\lambda^2}-\lambda^2\gamma\frac{2\lambda^2\alpha^2\gamma}{\beta^2+\alpha^2\lambda^2}+\lambda^2\gamma^2$$

$$=\frac{\cancel{\alpha^2\lambda^4\gamma^2}-\lambda^2\alpha^2\beta^2-\cancel{2\lambda^4\alpha^2\gamma^2}+\lambda^2\beta^2\gamma^2+\cancel{\alpha^2\lambda^4\gamma^2}}{\beta^2+\alpha^2\lambda^2}=\frac{\lambda^2\beta^2(\gamma^2-\alpha^2)}{\beta^2+\alpha^2\lambda^2}$$

$$=\frac{-\lambda^2\beta^4}{\beta^2+\alpha^2\lambda^2}$$

$$y_1+y_2=\lambda(x_1+x_2)-2\lambda\gamma=\lambda\frac{2\lambda^2\alpha^2\gamma}{\beta^2+\alpha^2\lambda^2}-2\lambda\gamma=\frac{\cancel{2\lambda^3\alpha^2\gamma}-2\lambda\beta^2\gamma-\cancel{2\lambda^3\alpha^2\gamma}}{\beta^2+\alpha^2\lambda^2}$$

$$=\frac{-2\lambda\beta^2\gamma}{\beta^2+\alpha^2\lambda^2}$$

$$M_0\left(\frac{\lambda^2\alpha^2\gamma}{\beta^2+\alpha^2\lambda^2}, \frac{-\lambda\beta^2\gamma}{\beta^2+\alpha^2\lambda^2}\right)$$

εφ: $\beta^2 x_1 x + \alpha^2 y_1 y - \alpha^2\beta^2 = 0 \Longrightarrow \lambda_{\varepsilon\varphi} = -\dfrac{\beta^2 x_1}{\alpha^2 y_1}$

$\Rightarrow \lambda_u = \dfrac{\alpha^2 y_1}{\beta^2 x_1}$

$y - y_1 = \dfrac{\alpha^2 y_1}{\beta^2 x_1}(x - x_1) \Leftrightarrow \beta^2 x_1 y - \beta^2 x_1 y_1 = \alpha^2 y_1 x - \alpha^2 x_1 y_1$

$$\begin{cases} \alpha^2 y_1 x - \beta^2 x_1 y = x_1 y_1 \gamma^2 \\ \alpha^2 y_2 x - \beta^2 x_2 y = x_2 y_2 \gamma^2 \end{cases}$$

$$D = \begin{vmatrix} \alpha^2 y_1 & -\beta^2 x_1 \\ \alpha^2 y_2 & -\beta^2 x_2 \end{vmatrix} = -\alpha^2\beta^2 x_2 y_1 + \alpha^2\beta^2 x_1 y_2 = \alpha^2\beta^2(x_1 y_2 - x_2 y_1)$$

$$D_x = \begin{vmatrix} x_1 y_1 \gamma^2 & -\beta^2 x_1 \\ x_2 y_2 \gamma^2 & -\beta^2 x_2 \end{vmatrix} = -x_1 x_2 \beta^2 \gamma^2 y_1 + \beta^2 x_1 x_2 y_2 \gamma^2 = \beta^2\gamma^2 x_1 x_2 (y_2 - y_1)$$

$$D_y = \begin{vmatrix} \alpha^2 y_1 & x_1 y_1 \gamma^2 \\ \alpha^2 y_2 & x_2 y_2 \gamma^2 \end{vmatrix} = \alpha^2\gamma^2 y_1 y_2 x_2 - \alpha^2\gamma^2 y_1 y_2 x_1 = \alpha^2\gamma^2 y_1 y_2 (x_2 - x_1)$$

$$P\left(\frac{\cancel{\beta^2}\gamma^2 x_1 x_2 (y_2 - y_1)}{\alpha^2\cancel{\beta^2}(x_1 y_2 - x_2 y_1)}, \frac{\cancel{\alpha^2}\gamma^2 y_1 y_2 (x_2 - x_1)}{\cancel{\alpha^2}\beta^2 (x_1 y_2 - x_2 y_1)}\right)$$

$$\lambda_{M_0P} = \frac{y_{M_0} - y_P}{x_{M_0} - x_P}$$. Προυκειμένου να δείξω ότι

$M_0P // x'x$ $\underline{\text{ΑΝΔ}}$ $y_{M_0} = y_P \Longrightarrow$

$$\frac{\cancel{\gamma^2} y_1 y_2 (x_2 - x_1)}{\beta^2 (x_1 y_2 - x_2 y_1)} = \frac{-\lambda \beta^2 \cancel{\gamma}}{\beta^2 + \alpha^2 \lambda^2} \Leftrightarrow$$

$$\frac{\cancel{-}\gamma \frac{\cancel{\lambda^2} \cancel{\beta^4}}{\cancel{\beta^2 + \alpha^2 \lambda^2}} (x_2 - x_1)}{\cancel{\beta^2} (x_1 y_2 - x_2 y_1)} = \frac{\cancel{-\lambda} \cancel{\beta^2}}{\cancel{\beta^2 + \alpha^2 \lambda^2}} \Leftrightarrow$$

$$\gamma \lambda (x_2 - x_1) = x_1 y_2 - x_2 y_1 \Leftrightarrow$$

$$\gamma \frac{y_2 - y_1}{\cancel{x_2 - x_1}} \cancel{(x_2 - x_1)} = x_1 y_2 - x_2 y_1 \Leftrightarrow$$

$$\gamma y_2 - \gamma y_1 = x_1 y_2 - x_2 y_1 .$$

Όμως $\vec{EM_1} // \vec{EM_2} \Leftrightarrow \begin{vmatrix} x_1 - \gamma & y_1 \\ x_2 - \gamma & y_2 \end{vmatrix} = 0 \Leftrightarrow$

$$x_1 y_2 - \gamma y_2 - x_2 y_1 + \gamma y_1 = 0 \Leftrightarrow$$

$$x_1 y_2 - x_2 y_1 = \gamma y_2 - \gamma y_1 .$$